Dai-Freed Anomalies and Cobordisms

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Anomalies in QFT

- An anomaly is a classical symmetry that fails to survive quantization.
- In QFT, given a field transformation, an anomaly signals that the following implication does not hold in general:

$$S' = S \quad \underset{\text{not necessarily!}}{\Longrightarrow} \quad Z' = Z. \tag{1}$$

- How is this possible?

Why Anomalies Exist

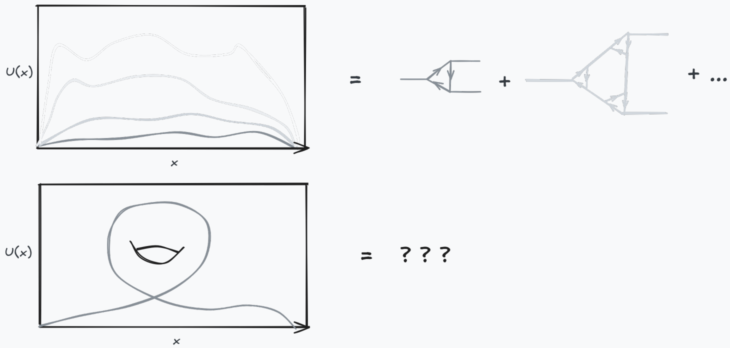
$$Z = \int \mathcal{D}\phi \, e^{iS[\phi]} \tag{2}$$

- If $\phi \to \phi'$ is a symmetry of the classical theory, then $S[\phi'] = S[\phi]$.
- However, Z does not depend on ϕ through $S[\phi]$ alone, ϕ also appears in the integration measure $\mathcal{D}\phi$.
- Is $\mathcal{D}\phi' = \mathcal{D}\phi$? Under a change of variables, the most general relation involves a Jacobian J:

$$\mathcal{D}\phi' = J\mathcal{D}\phi. \tag{3}$$

- $J \neq 1$ generically in systems with massless fermions coupled to gauge fields or to gravity.

Perturbative and Non-Perturbative Anomalies

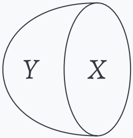


- Anomalous transformations $U(x^\mu)$ connected to the identity $U_{\text{id}}(x^\mu) = 1$ are detected by perturbative methods.
- If the symmetry group G, viewed as a manifold, has nontrivial topology, there exist families of transformations not connected to the identity.

Theorem (Dai-Freed)

If a fermionic theory in D-dimensional spacetime can be realized as the boundary of a $(D+1)$-dimensional theory, then there exists a well-defined phase

$$Z = |Z| e^{2\pi i \eta}. \tag{4}$$



- η is an invariant depending only on the bulk extension Y. It is defined from the spectrum of $i\not{D}_Y$:

$$\eta = \frac{1}{2}\left(\#\text{positive} - \#\text{negative} + \#\text{zero}\right). \tag{5}$$

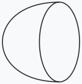
- In essence, the Dai-Freed theorem assigns a phase to each manifold Y. It can be viewed as a map

$$\text{Manifolds} \to U(1). \tag{6}$$

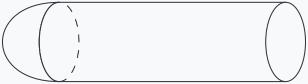
- In practice, this must be refined: requiring the existence of fermions and other fields restricts the class of manifolds under consideration to those carrying appropriate structures.

A Visual Perspective

- A fermionic theory on X:



- Applying a transformation:



- This cylinder from X to X represents some transformation $T : \mathcal{H}_X \to \mathcal{H}_X$ of the theory[1].
- By itself, T defines another bulk extension. If the two extensions yield different phases, an anomaly is present.

[1] Example: $A^\mu(t) = (1-t)A^\mu + t(A')^\mu$

$Y =$

- Two extensions Y_1 and Y_2 can be glued together, after reversing the orientation of one, to form a closed manifold Y:

$$e^{2\pi i \eta} = e^{2\pi i(\eta_1 - \eta_2)}. \tag{7}$$

- The anomaly-cancellation condition then becomes

$$e^{2\pi i \eta} = 1 \tag{8}$$

for every closed manifold Y that contains the original theory (on X) as a slice.

Local Anomalies Revisited

- η satisfies the Atiyah-Patodi-Singer (APS) index theorem, that is, a relation between the fermionic spectrum and the geometry and topology of spacetime:

$$\eta = \underbrace{\mathrm{index}\, i\not{D}_Z}_{\text{integer}} - \overbrace{\int_Z \underbrace{\widehat{A}(R)}_{\text{curvature}} \wedge \underbrace{\mathrm{ch}(F)}_{\text{gauge fields}}}^{\text{depends on the bulk}} . \qquad (9)$$

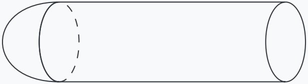
- Since $e^{2\pi i \eta} = e^{-2\pi i \int \widehat{A}\,\mathrm{ch}}$, anomaly cancellation reduces to requiring $\widehat{A} \wedge \mathrm{ch} = 0$.

Global Anomalies Revisited

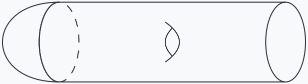
- In the absence of local anomalies, only the boundary data matters. Families of $(D+1)$-dimensional manifolds connected by a $(D+2)$-dimensional cobordism yield the same phase. The phase thus defines a map
$$\Omega_{D+1}^{\text{special}} \to U(1). \tag{10}$$
- This is in fact a group homomorphism, and therefore maps the identity to the identity.
- Consequently, if $\Omega_{D+1}^{\text{special}}$ is trivial, then $e^{2\pi i\eta} = 1$ automatically.
- The condition $\Omega_{D+1}^{\text{special}} = 0$ is thus the criterion for the absence of global anomalies. It can be computed algebraically using spectral sequences.

Closing Remarks

- The manifolds Y we consider are sufficiently general that they do not merely correspond to changes in the field configuration.



- They also allow for changes in the topology of spacetime (changes in the number of handles or connected components).



- The Dai-Freed theorem is therefore sensitive not only to traditional anomalies, but also to a broader class of anomalies that are natural in theories where topology change is permitted.

Summary:

- The Dai-Freed theorem states that if X "=" ∂Y, then

$$Z = |Z| e^{2\pi i \eta}.$$

- It detects both traditional anomalies and a broader class of generalized ones.
- If $\widehat{A} \wedge \mathrm{ch}\,|_{D+2} \neq 0$, perturbative (local) anomalies are present.
- If $\Omega_{D+1}^{\text{special}} \neq 0$, non-perturbative (global) anomalies are present.
- Anomalies are ultimately a consequence of the geometry and topology of spacetime, encoded in the quantities $\widehat{A},\ \mathrm{ch},\ \Omega,\ \eta$, etc.

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Thank you for your attention.